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International Journal of Heat and Mass Transfer 51 (2008) 1393–1401
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Pressure drop predictions for laminar flows of extended modified power law fluids in rectangular ducts

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Received 14 March 2007; received in revised form 21 September 2007
Available online 28 January 2008

Abstract

This paper reports on a numerical study of the friction factor-modified Reynolds number product, $f \times Re_M$, for fully developed, laminar flows of pseudoplastic and dilatant fluids in rectangular ducts. Constitutive equations for the apparent viscosity that span from the low shear rate Newtonian region through the high shear rate Newtonian region were utilized, and a shear rate parameter was defined that determines the flow regime where the duct is operating. Numerical results for $f \times Re_M$ in all flow regimes are included along with correlation equations. Errors associated with applying power law solutions to flows of weakly non-Newtonian fluids are discussed.

Keywords: Friction factor; Modified Reynolds number; Extended modified power law; Rectangular duct flow; Non-Newtonian fluids

1. Introduction

Pseudoplastic and dilatant fluids are non-Newtonian fluids that are purely viscous, time-independent, and that exhibit no yield stress. Their apparent viscosity, η_a, is a function of shear rate, $\dot{\gamma}$, and for pseudoplastic (i.e., shear-thinning) fluids, decreases with increasing shear rate from a maximum value of η_0 (the zero shear rate viscosity) to a minimum value of η_∞ (the infinite shear rate viscosity) as shown pictorially in Fig. 1. Five distinct regions may be defined: A low shear rate Newtonian region, Region I; a low shear rate transition region, Region II; a power law region, Region III; a high shear rate transition region, Region IV; and a high shear rate Newtonian region, Region V. Dilatant (i.e., shear thickening) fluids behave in a similar fashion except that the slope in Region III is positive so that $\eta_\infty > \eta_0$. Since any given flow field may contain shear rates in more than one of the above regions, it is important when solving flow problems to utilize constitutive equations that span the entire shear rate range. Solutions may otherwise be inapplicable to the physical flow conditions. For example, the power law model,

$$\eta_{aP} = K\dot{\gamma}^{n-1} \tag{1}$$

where η_{aP} is the power law apparent viscosity, K is the fluid consistency, and n is the flow index ($n < 1$ for pseudoplastic fluids, $n > 1$ for dilatant fluids, and $n = 1$ for Newtonian fluids), overestimates the apparent viscosity of pseudoplastic fluids in the areas of the flow field where the shear rates are in Regions I and II and underestimates it where the shear rates are in Regions IV and V (the opposite is true for dilatant fluids). Furthermore, it predicts an infinite apparent viscosity wherever the shear rate goes to zero, for example, along the centerline of an axisymmetric duct. Nevertheless, an extensive set of solutions that utilize the power law model is available in the literature. But care must be exercised when applying these solutions to ensure that the shear rates throughout the flow field are in Region III so that the solutions apply.

Models that span the entire shear rate range have been defined. However, these generally contain one or more fluid properties that differ from η_0, η_∞, K, and n. This complicates the comparison of results obtained using such models

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doi:10.1016/j.ijheatmasstransfer.2007.11.019

Nomenclature

A	duct cross-sectional area
a	duct half-height
b	duct half-width
d_H	hydraulic diameter
f	Darcy friction factor
K	power law fluid consistency
m	fluid parameter in Cross model
n	flow index
P	pressure
Re	Newtonian Reynolds number based on η_N
Re_g	generalized Reynolds number
Re_M	modified Reynolds number
Re_0	Newtonian Reynolds number based on η_0
Re_∞	Newtonian Reynolds number based on η_∞
u, v, w	velocity components in the x, y, and z directions, respectively
$\bar{u}$	bulk fluid velocity
x, y, z	coordinate directions

Greek symbols

α^*	duct width-to-height aspect ratio
β_0	MPL shear rate parameter based on η_0
β_∞	MPL shear rate parameter based on η_∞
β^*	EMPL shear rate parameter
$\dot{\gamma}$	shear rate
η_a	apparent viscosity
η_{aP}	power law apparent viscosity
η_N	Newtonian fluid viscosity
η_P	power law reference viscosity
η_0	zero shear rate viscosity
η_∞	infinite shear rate viscosity
η^*	EMPL reference viscosity
η^*_{MPL}	MPL reference viscosity
λ	fluid parameter in Cross model
ρ	fluid density

Subscripts

yx	x-direction quantity on a y-plane
zx	x-direction quantity on a z-plane

Superscript

$+$	refers to a dimensionless quantity

to existing results generated with the power law model in those cases where the latter solution applies. For example, Cross [1] proposed the following model for pseudoplastic fluids:

$$\eta_a = \left(\frac{\eta_0 - \eta_\infty}{1 + \lambda \cdot \dot{\gamma}^m}\right) + \eta_\infty \tag{2}$$

Here, m is a dimensionless fluid property, and λ is a parameter such that $\lambda^{-1/m}$ is that shear rate where the apparent viscosity attains a value equal to the average of the limiting viscosities, η_0 and η_∞. Thus, the flow curve is defined by four properties, but the power law properties, K and n, are absent from this model and the properties m and λ appear instead.

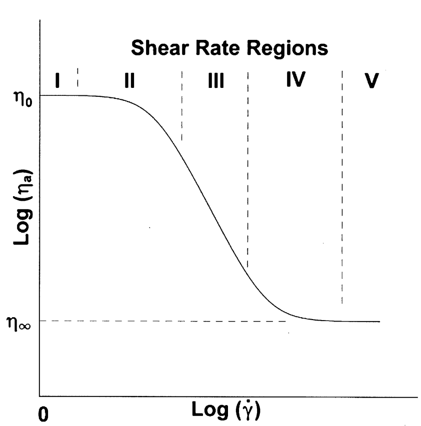

Fig. 1. Typical flow curve of a pseudoplastic (i.e., shear-thinning) fluid: Region I – low shear rate Newtonian region; Region II – low shear rate transition region; Region III – power law region; Region IV – high shear rate transition region; Region V – high shear rate Newtonian region.

Dunleavy and Middleman [2] proposed a model for the flow curve of pseudoplastic fluids in Regions I through III which maintains the power law parameters K and n:

$$\eta_a = \frac{\eta_0}{1 + \frac{\eta_0}{\eta_{aP}}} = \frac{\eta_0}{1 + \frac{\eta_0}{K} \cdot \dot{\gamma}^{1-n}} \tag{3}$$

This model, termed the modified power law (MPL) model, provides a flow curve that approaches the low shear rate Newtonian viscosity as the shear rate approaches 0 and the power law apparent viscosity when the shear rate becomes sufficiently large. A smooth transition occurs between these two regions. However, since Regions IV and V are missing from this model, care must be exercised when utilizing Eq. (3) in those instances where the flow field contains shear rates that exceed those of Region III.

The MPL model has been used to solve several problems of engineering interest, including the friction factor-modified Reynolds number product, $f \times Re_M$, in hydrodynamically fully developed, laminar duct flows. For example, Brewster and Irvine [3] solved the circular duct problem, Capobianchi and Irvine [4] calculated the solution for annular ducts, and Park et al. [5] evaluated the values for rectangular ducts. The latter study includes experimental data obtained utilizing three different aqueous solutions of sodium carboxymethyl cellulose and shows good agreement between the measured and numerical values, with the experimental data falling within ±4.9% of the numerical results.

Brewster and Irvine [3] defined a shear rate parameter, β_0, whose value determines where the duct is operating

$$\beta_0 = \frac{\eta_0}{\eta_P} = \frac{Re_g}{Re_0} : \eta_P = \frac{K}{(\bar{u}/d_H)^{1-n}} : Re_0 = \frac{\rho \bar{u} d_H}{\eta_0} :$$
$$Re_g = \frac{\rho \bar{u} d_H}{\eta_P} \tag{4}$$

Here, η_P is the power law reference viscosity, $\bar{u}$ is the bulk fluid velocity, d_H is the hydraulic diameter, ρ is the fluid density, and Re_0 and Re_g are the Newtonian and the generalized (i.e., power law) Reynolds numbers, respectively. Furthermore, a modified Reynolds number, Re_M, was introduced given by

$$Re_M = \frac{\rho \bar{u} d_H}{\eta^*_{MPL}} = Re_0 + Re_g : \eta^*_{MPL} = \frac{\eta_0}{1+\beta_0} \tag{5}$$

where η^*_{MPL} is a reference viscosity. For a given duct, small values of β_0 indicate that the duct is operating in the low shear rate Newtonian region so that $f \times Re_M$ and η^*_{MPL} approach their respective Newtonian values. If β_0 is sufficiently large, then the duct is operating in the power law region and $f \times Re_M$ and η^*_{MPL} approach the power law values. At intermediate values of β_0, the duct is operating in the low shear rate transition region. Then $f \times Re_M$ and η^*_{MPL} attain values between the Newtonian and the power law values, their magnitudes dependent on the value of β_0.

The Capobianchi and Irvine [4] study proposed an analogous MPL model for dilatant fluids that provides a low shear rate transition region by blending the apparent viscosities of Regions I and III in the same manner as in the MPL expression for pseudoplastic fluids

$$\eta_a = \eta_0 \cdot \left(1 + \frac{\eta_{aP}}{\eta_0}\right) = \eta_0 \cdot \left(1 + \frac{K}{\eta_0} \cdot \dot{\gamma}^{n-1}\right) \tag{6}$$

The same definitions for β_0 and Re_M were used except that η^*_{MPL} was redefined as

$$\eta^*_{MPL} = \eta_0 \left(1 + \frac{1}{\beta_0}\right) \tag{7}$$

The values of $f \times Re_M$ and η^*_{MPL} again depend on the value of β_0.

The present study calculates $f \times Re_M$ for hydrodynamically fully developed laminar flows of pseudoplastic and dilatant fluids in rectangular ducts. In contrast to earlier studies, it utilizes constitutive equations that span the entire shear rate range so that the results are applicable for all shear rate conditions that may exist in the duct. These are derived from the MPL expressions, Eqs. (3) and (6), and contain only the fluid properties η_0, η_∞, K, and n. A shear rate parameter and reference viscosities are defined that serve the analogous purposes as the corresponding parameters used with the MPL models. The solutions are a function of this shear rate parameter and have values that range from the Newtonian value when the shear rate parameter is either in the low or high shear rate Newtonian regions, to the power law value when the shear rate parameter is in the power law region, to intermediate values when the duct is operating in either of the transition regions, Regions II or IV. It should be noted that the current results cannot be extracted by simply re-parametrizing existing results because the latter are generated using constitutive equations that are valid over only a portion of the shear rate range. Furthermore, the $f \times Re_M$ values for ducts operating in Region IV are particularly significant since there appears to be no data in the literature in this regime. Finally, the results show that significant errors occur when applying the power law solution to flows of weakly non-Newtonian fluids (i.e., fluids for which η_0 and η_∞ are of the same order) even for cases where the shear rates are everywhere in Region III.

2. Analysis

2.1. *Definition of the extended modified power law (EMPL) constitutive equations*

The goal is to define constitutive equations for pseudoplastic and dilatant fluids that span the entire shear rate range while containing only the properties η_0, η_∞, K, and n. For pseudoplastic fluids, a constitutive equation that meets these requirements may be generated by combining the MPL models, Eqs. (3) and (6), as follows. First, it may be observed that if η_∞ replaces η_0 in Eq. (6), the resulting expression is suitable to describe the flow curve of pseudoplastic fluids in Regions III, IV, and V. If this expression is then added to the MPL equation for pseudoplastic fluids, Eq. (3), the flow curve given by the resulting sum has the proper shape but everywhere overestimates the apparent viscosity by the power law apparent viscosity, η_{aP}. Thus, subtracting η_{aP} from this sum gives

$$\eta_a = \frac{\eta_0}{1 + \frac{\eta_0}{\eta_{aP}}} + \eta_\infty \tag{8}$$

But when $\dot{\gamma}$ approaches 0 in Eq. (8), η_a approaches $\eta_0 + \eta_\infty$ rather than η_0. This is corrected by subtracting η_∞ from the numerator of the first term on the right hand side, giving

$$\eta_a = \frac{\eta_0 - \eta_\infty}{1 + \frac{\eta_0}{\eta_{aP}}} + \eta_\infty = \frac{\eta_0 - \eta_\infty}{1 + \frac{\eta_0}{K} \cdot \dot{\gamma}^{1-n}} + \eta_\infty \tag{9}$$

A model for dilatant fluids may be constructed by applying the same procedure except that η_∞ replaces η_0 in Eq. (3) rather than in Eq. (6), and that the resulting expression is added to Eq. (6). Also, the final correction is achieved by subtracting η_0 rather than η_∞. The resulting model for dilatant fluids is then

$$\eta_a = \frac{\eta_\infty - \eta_0}{1 + \frac{\eta_\infty}{\eta_{aP}}} + \eta_0 = \frac{\eta_\infty - \eta_0}{1 + \frac{\eta_\infty}{K} \cdot \dot{\gamma}^{1-n}} + \eta_0 \tag{10}$$

Since Eqs. (9) and (10) were derived from the MPL expressions, and because they extend the applicable range of the MPL models to the entire shear rate regime, they are termed the extended modified power law (EMPL) models. Furthermore, the first consideration implies that the transition regions, Regions II and IV, are generated by the same physics that led to the MPL models. It may be shown that the apparent viscosity predicted by the EMPL equations approaches the MPL expressions when the shear rates are between the power law and Newtonian regions, provided that η_0 and η_∞ differ sufficiently from each other. However, for weakly non-Newtonian fluids, a true power law region fails to become completely established because the flow curve begins its transition to the high shear rate Newtonian region before the low shear rate transition region is able to complete its transformation to power law behavior. The EMPL models then blend Regions II, III, and IV into a smooth, continuous transition region in the same fashion as the Cross and other models that are valid for the entire shear rate range. As will be shown, the consequence for weakly non-Newtonian fluids is that solutions that utilize the power law model are inapplicable even when the shear rate in the entire flow field is in Region III.

For pseudoplastic fluids, the similarity of the EMPL model, Eq. (9), to the Cross model, Eq. (2), should be noted, and these become equivalent if $m = 1 - n$ and $\lambda = \eta_0/K$. The EMPL model was used instead of the Cross model because the latter as originally proposed fails to satisfy the goal of having only the fluid properties η_0, η_∞, K, and n in the definition, and because no equivalent Cross model was given for dilatant fluids. Finally, for Newtonian fluids, the EMPL models predict a constant value for the apparent viscosity as may be verified by setting η_0 and η_∞ equal to the Newtonian viscosity, η_N.

2.2. Analysis of the rectangular duct problem

The EMPL models were used to solve the $f \times Re_M$ problem in rectangular ducts. The flows considered are hydrodynamically fully developed, incompressible, laminar flows of pseudoplastic and dilatant fluids in rectangular ducts with negligible viscous dissipation. All fluid properties are assumed constant except for viscosity which is assumed to be a function of shear rate according to the EMPL models. The duct, shown schematically in Fig. 2, has a height of $2a$ and a width of $2b$, and has coordinate axes located at its center with the x-axis aligned along the longitudinal (i.e., flow) direction and the y and z axes parallel to the height and width of the duct, respectively. The governing momentum equation for this case is then

$$\frac{\partial}{\partial y}\left(\eta_{a,yx}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta_{a,zx}\frac{\partial u}{\partial z}\right) = \frac{dP}{dx} \tag{11}$$

where P is pressure and the subscripts yx and zx in the apparent viscosity terms are coordinate direction pairs that indicate the plane and direction, respectively, of the shear rate in the apparent viscosity expressions. Because of symmetry and of the no-slip condition, the appropriate boundary conditions are

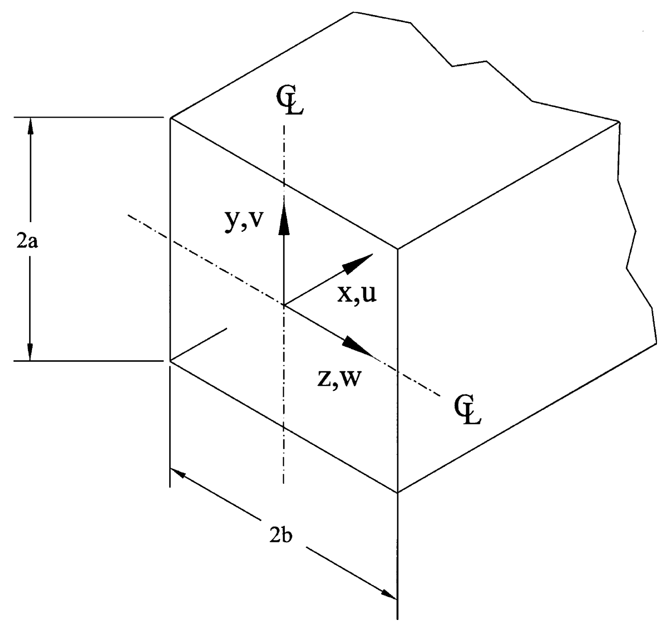

Fig. 2. Geometry of the rectangular duct problem.

$$\left(\frac{\partial u}{\partial z}\right)_{(y,z=0)} = 0 : \left(\frac{\partial u}{\partial y}\right)_{(y=0,z)} = 0$$
$$u(y, z = b) = 0 : u(y = a, z) = 0 \tag{12}$$

Furthermore, the global continuity equation is given by

$$\bar{u} = \frac{1}{A}\int_A u\,dA = \frac{1}{ab}\int_{y=0}^{a}\int_{z=0}^{b} u(y,z)\,dz\,dy \tag{13}$$

where A is the cross-sectional area of the duct.

For incompressible, purely viscous, non-Newtonian fluids in simple shear where the only non-zero velocity component is u (function of y and z only), the shear rates in the expressions for $\eta_{a,yx}$ and $\eta_{a,zx}$ in Eq. (11) are equal and are given by (see Hartnett and Kostic [6] for a more complete discussion)

$$\dot{\gamma} = \dot{\gamma}_{yx} = \dot{\gamma}_{zx} = \left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right\}^{1/2} \tag{14}$$

To evaluate $f \times Re_M$ for the above problem, Eqs. (11)–(14) are first non-dimensionalized using the following definitions:

$$y^+ = \frac{y}{d_H} : z^+ = \frac{z}{d_H} : u^+ = \frac{u}{\bar{u}} : \dot{\gamma}^+ = \frac{\dot{\gamma}}{\bar{u}/d_H} : \eta_a^+ = \frac{\eta_a}{\eta^*}$$
$$\alpha^* = \frac{b}{a} : f = \frac{-(dP/dx)d_H}{\rho\bar{u}^2/2} : Re_M = \frac{\rho\bar{u}d_H}{\eta^*} \tag{15}$$

where f is the Darcy friction factor, α^* is the duct aspect ratio, and η^* is a reference viscosity, defined later, that is specific to the EMPL models and that differs from the formulations used with the MPL models. Substituting Eqs. (15) into Eq. (11) yields

$$\frac{\partial}{\partial y^+}\left(\eta_a^+\frac{\partial u^+}{\partial y^+}\right)+\frac{\partial}{\partial z^+}\left(\eta_a^+\frac{\partial u^+}{\partial z^+}\right)=-\frac{f\cdot Re_M}{2} \tag{16}$$

with boundary conditions, Eqs. (12), becoming

$$\left(\frac{\partial u^+}{\partial z^+}\right)_{(y^+,z^+=0)}=0:\left(\frac{\partial u^+}{\partial y^+}\right)_{(y^+=0,z^+)}=0$$
$$u^+\left(y^+,z^+=\frac{1+\alpha^*}{4}\right)=0:u^+\left(y^+=\frac{1+\alpha^*}{4\alpha^*},z^+\right)=0 \tag{17}$$

The global continuity equation, Eq. (13), is also cast in dimensionless form giving

$$1=\frac{16\cdot\alpha^*}{(1+\alpha^*)^2}\int_{y^+=0}^{\frac{1+\alpha^*}{4\cdot\alpha^*}}\int_{z^+=0}^{\frac{1+\alpha^*}{4}}u^+\,dz^+\,dy^+ \tag{18}$$

Finally, the shear rate, Eq. (14), in dimensionless form becomes

$$\dot{\gamma}^+=\left\{\left(\frac{\partial u^+}{\partial y^+}\right)^2+\left(\frac{\partial u^+}{\partial z^+}\right)^2\right\}^{1/2} \tag{19}$$

Next, a reference viscosity, η^*, must be defined for use with the EMPL models. The goal is to choose a functional form that approaches the appropriate value when the duct is operating in any of the five regions of the flow curve. For pseudoplastic fluids, a reference viscosity that meets the above requirement is

$$\eta^*=\frac{\eta_0-\eta_\infty}{1+\frac{\eta_0}{\eta_P}}+\eta_\infty \tag{20}$$

Eq. (20) may be written in terms of the MPL shear rate parameter, β_0, defined in Eq. (4), and of an analogous parameter, β_∞, defined as

$$\beta_\infty=\frac{\eta_\infty}{\eta_P}=\frac{Re_g}{Re_\infty} \tag{21}$$

where Re_∞ is the Newtonian Reynolds number based on η_∞. Eq. (20) then becomes

$$\eta^*=\left[\frac{1+\beta_\infty}{1+\beta_0}\right]\times\eta_0 \tag{22}$$

Similarly, for dilatant fluids η^* may be defined as

$$\eta^*=\frac{\eta_\infty-\eta_0}{1+\frac{\eta_\infty}{\eta_P}}+\eta_0=\left[\frac{1+\beta_0}{1+\beta_\infty}\right]\times\eta_\infty \tag{23}$$

The terms in brackets in Eqs. (22) and (23) may be defined as a parameter, β^*

$$\beta^*=\frac{1+\beta_0}{1+\beta_\infty}=\frac{\beta_0}{\beta_\infty}\times\frac{Re_0+Re_g}{Re_\infty+Re_g} \tag{24}$$

where the coefficient β_0/β_∞ evaluates to the limiting viscosity ratio, η_0/η_∞, and the remainder of the right hand side is a function of the limiting Reynolds numbers. The definitions of the reference viscosities then become

$$n\leqslant 1:\eta^*=\frac{\eta_0}{\beta^*}:n\geqslant 1:\eta^*=\beta^*\times\eta_\infty \tag{25}$$

Eqs. (25) may be shown to approach the correct apparent viscosity in all regions, with the value in the transition regions equal to the MPL reference viscosities. Thus β^* is the appropriate shear rate parameter for EMPL models and its value determines where the duct is operating. That is, β^* performs the same role for solutions that use the EMPL models that β_0 performs for solutions that use the MPL models.

For pseudoplastic fluids, β^* ranges between 1 and β_0/β_∞, the endpoints indicating operation in the low shear rate and the high shear rate Newtonian regions, respectively. The power law solution is reached if η_0 differs sufficiently from η_∞ in which case the viscosity in the power law region is such that $\eta_0\gg\eta_P\gg\eta_\infty$. This leads to $\beta_0\gg 1$ and $\beta_\infty\ll 1$ so that β^* approaches the value of β_0. Then, for a fixed value of β_0/β_∞, β^* may be found by substituting β^* for β_0 and $\beta^*\times(\beta_\infty/\beta_0)$ for β_∞ in the right hand side of Eq. (24), giving

$$\beta^*=\sqrt{\beta_0/\beta_\infty}\Rightarrow\log\beta^*=\frac{1}{2}\log(\beta_0/\beta_\infty) \tag{26}$$

That is, the power law value occurs at the midpoint of the $\log(\beta^*)$ range. Analogous arguments and conclusions apply for dilatant fluids, noting that there the valid range of β^* is reversed (i.e., β_0/β_∞ to 1).

With the shear rate parameter defined, the EMPL models, Eqs. (9) and (10), may be cast in dimensionless form in terms of β^*, β_0, β_∞, and $\dot{\gamma}^+$ giving

$$n\leqslant 1:\eta_a^+=\beta^*\times\left\{\frac{1+\beta_\infty\cdot(\dot{\gamma}^+)^{1-n}}{1+\beta_0\cdot(\dot{\gamma}^+)^{1-n}}\right\}:$$
$$n\geqslant 1:\eta_a^+=\frac{1}{\beta^*}\times\left\{\frac{1+\beta_0\cdot(\dot{\gamma}^+)^{1-n}}{1+\beta_\infty\cdot(\dot{\gamma}^+)^{1-n}}\right\} \tag{27}$$

Thus, the apparent viscosities in dimensionless form are the reciprocals of each other.

The dimensionless momentum equation with boundary conditions, Eqs. (16) and (17), may then be solved simultaneously with the dimensionless global continuity equation, Eq. (18), utilizing Eq. (27) to evaluate the dimensionless apparent viscosity depending on the fluid type. This results in the evaluation of $f\times Re_M$. Four parameters remain: The duct aspect ratio, α^*, the flow index, n, the shear rate parameter, β^*, and the limiting viscosity ratio, β_0/β_∞. It should be noted that choosing β^* and β_0/β_∞ fixes the values of β_0 and β_∞ via Eq. (24) as detailed in the algorithm below. The section that follows describes the numerical solution to the above equations and presents the results along with a discussion.

3. Results and discussion

3.1. The numerical scheme

The problem was discretized and solved numerically using a finite volume scheme. A uniform rectangular mesh was utilized with zero-thickness control volumes at the

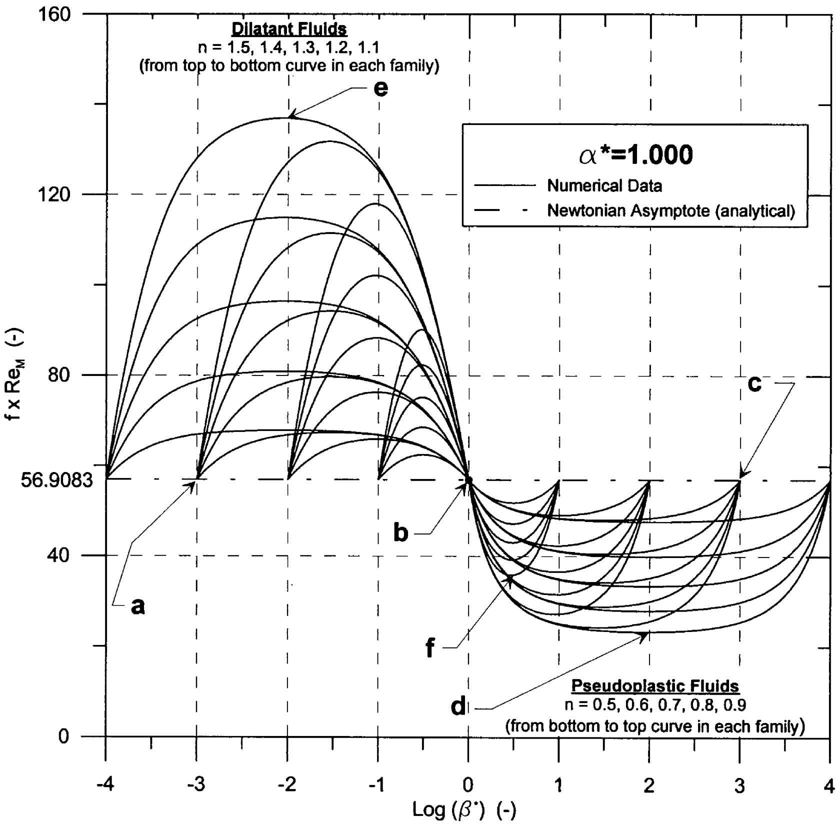

Fig. 3. $f \times Re_M$ solution for $\alpha^* = 1.000$. **a** – low shear rate Newtonian value for dilatant fluids with $\beta_0/\beta_\infty = 10^{-3}$, **b** – high shear rate Newtonian value for dilatant fluids, low shear rate Newtonian value for pseudoplastic fluids, and solution for Newtonian ($n = 1$) fluids, **c** – high shear rate Newtonian value for pseudoplastic fluids with $\beta_0/\beta_\infty = 10^3$, **d** – power law value for $n = 0.5$, **e** – power law value for $n = 1.5$, and **f** – minimum value of $f \times Re_M$ for a pseudoplastic fluid with $\beta_0/\beta_\infty = 10$ and $n = 0.5$.

boundaries of the domain. The global continuity equation, Eq. (18), was discretized using an area integral formulation of the trapezoidal rule. The resulting set of algebraic equations was solved using the LSLXG routine (in Version 5.0 of the IMSL Fortran Numerical Library, Visual Numerics, Inc.), a sparse matrix routine that utilizes Gaussian elimination to solve systems of simultaneous equations. The solution then proceeded as follows:

1. Select run parameters (α^*, β_0/β_∞, n, β^*, mesh density, convergence criteria, etc.).
2. For each value of β^*:
   a. Use the velocity distribution, $u^+(y^+,z^+)$, from the previous, converged β^* data point to initialize the velocity distribution for the current value of β^*.
   b. Calculate the value of β_∞ from Eq. (24) by substituting $\beta_\infty \times (\beta_0/\beta_\infty)$ for β_0 and solving for β_∞. Then calculate the value of β_0 from $\beta_0 = \beta_\infty \times (\beta_0/\beta_\infty)$.
   c. Calculate η_a^* using Eq. (27) and update the coefficient matrix.
   d. Solve the system of algebraic equations as described above.
   e. Check for convergence at each node. If converged, then store the data and proceed to Step 3. Otherwise, return to Step 2c using the latest velocity values.
3. Next value of β^*.

The above algorithm was used to calculate the values of $f \times Re_M$, reaching converged values generally within ten iterations per data point. Mesh refinement studies showed that solutions became practically independent of mesh density for densities of 100 × 100 nodes, with increases in mesh densities of 50% in each direction generally yielding changes of 0.02% or less. The exception was the parallel-plate duct which required a 350 × 350 mesh to achieve results that were practically independent of mesh density.

To validate the program, calculated values were compared to the numerical results of Park et al. [5], Syrjälä [7], Gao and Hartnett [8], Wheeler and Wissler [9], Chandrupatla [10][1], and Schechter [11][1]. There was good agreement throughout, generally within 0.5%, except for the Park et al. study where the present data agreed within 4.1%. However, the Park et al. study used an approximate expression for the shear rate that eliminated the cross coordinate term in Eq. (19). This simplification accounts for their values being higher than those from this and the other studies. Furthermore, computed data were compared to available analytical solutions: the Newtonian $f \times Re$ solutions for rectangular and parallel-plate ducts, and the power law $f \times Re_g$ solution for parallel-plate ducts. For

[1] Data as reported by Hartnett and Kostic [6].

rectangular ducts, all Newtonian values were well within 0.01% of the corresponding analytical values, and they were within 0.07% for the parallel-plate duct. The power law values for the parallel-plate duct were all within 0.3% of the analytical solutions. With the program thus validated, the rectangular duct problem was solved.

3.2. Numerical solution of the rectangular duct problem

The $f \times Re_M$ problem was solved numerically for rectangular ducts of five aspects ratios, $\alpha^* = 1.00$, 0.75, 0.50, 0.25, and $\to\infty$ (the latter calculated with $\alpha^* = 10^5$), with computations performed for both dilatant and pseudoplastic fluids. For dilatant fluids, values of 10^{-4}, 10^{-3}, 10^{-2}, and 10^{-1} were selected for β_0/β_∞ and for each, solutions were calculated for $n = 1.1$, 1.2, 1.3, 1.4, and 1.5. Similarly, solutions for pseudoplastic fluids were obtained for $\beta_0/\beta_\infty = 10^1$, 10^2, 10^3, and 10^4 and $n = 0.5$, 0.6, 0.7, 0.8, and 0.9. Thus, 40 curves were developed for each aspect ratio, each curve spanning the full range of the shear rate parameter β^*, and each comprised of 100 equally spaced, computed data points. The solutions for $\alpha^* = 1.00$ and $\alpha^* \to \infty$ are presented in Figs. 3 and 4, respectively. The solid curves are the numerical data computed in this study whereas the analytical Newtonian values are denoted by the phantom lines. Referring to Fig. 3, it should be noted that, as expected, the solutions for dilatant fluids reside above the Newtonian line at values of $\log(\beta^*)$ less than 0, and that those for pseudoplastic fluids are below the Newtonian line at values of $\log(\beta^*)$ greater than 0. Furthermore, the solutions are grouped in families of curves with each family being identified by the non-zero value of $\log(\beta^*)$ where the curves intersect the Newtonian line, with that value corresponding to the limiting viscosity ratio, β_0/β_∞. For example, all curves that intersect the Newtonian line at Point "a" in Fig. 3 belong to dilatant fluids with a limiting viscosity ratio of $\beta_0/\beta_\infty = 10^{-3}$, whereas those that intersect at Point "c" belong to pseudoplastic fluids with a limiting viscosity ratio of $\beta_0/\beta_\infty = 10^3$. All curves begin and end at the Newtonian line as predicted, with the left-most point of intersection corresponding to ducts operating in the low shear rate Newtonian region and the right-most intersection corresponding to those operating in the high shear rate Newtonian region. Thus, Point "b" in Fig. 3 is for duct flows of dilatant fluids operating in the high shear rate Newtonian region as well as for duct flows of pseudoplastic fluids operating in the low shear rate Newtonian region. The curve for Newtonian fluids (i.e., $n = 1$) degenerates to a single point located on the Newtonian line at $\log(\beta^*) = 0$ (i.e., Point "b" in Fig. 3) because for Newtonian fluids, $\eta_0 = \eta_\infty = \eta_N$ thus giving $\beta_0/\beta_\infty = 1$. The valid β^* range then collapses to a point located at $\beta^* = 1$.

Each curve has an extremum; points to the left of the extremum are the values of $f \times Re_M$ in the low shear rate transition region, and those to the right are the values of

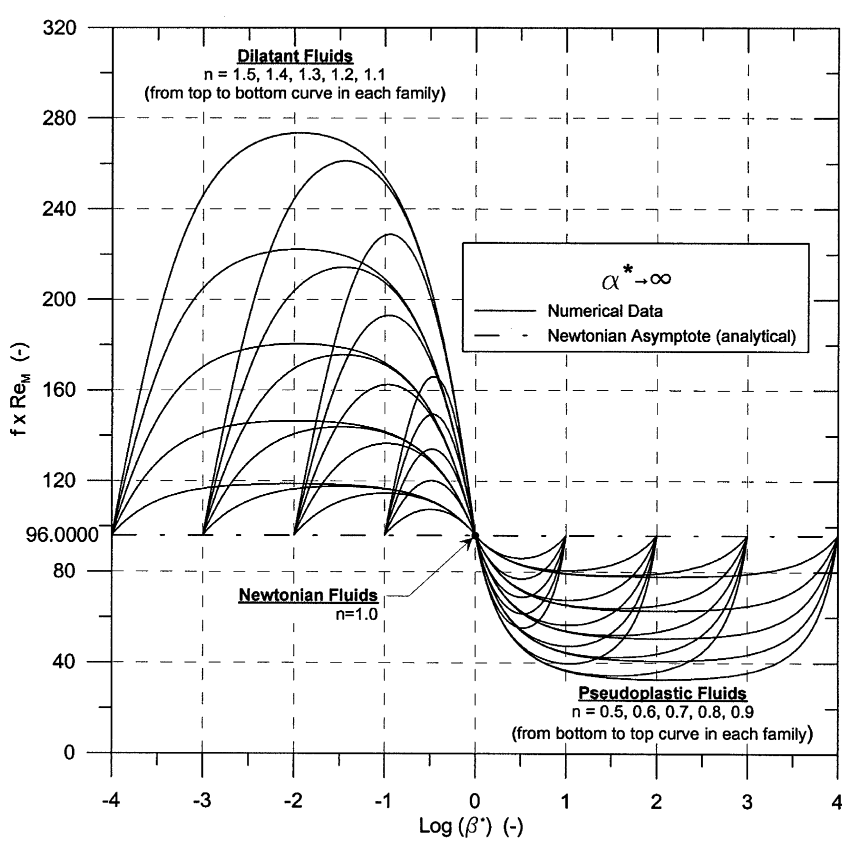

Fig. 4. $f \times Re_M$ solution for $\alpha^* \to \infty$.

Table 1
Correlation equation and respective constants for evaluating $f \times Re_M$

α^*	$f \times Re$	$f \times Re_g$ values at $n =$				
		0.5	0.6	0.7	0.8	0.9
Correlation equation for pseudoplastic fluids $f \cdot Re_M = \beta^* \times (f \cdot Re) \times \left\{ \frac{1 + \beta_\infty \times (f \cdot Re/f \cdot Re_g)}{1 + \beta_0 \times (f \cdot Re/f \cdot Re_g)} \right\}$						
1	56.9083	23.3302	27.9481	33.4295	39.9434	47.6903
0.75	57.9028	23.5505	28.2516	33.8439	40.5039	48.4405
0.5	62.1922	24.5129	29.5800	35.6552	42.9445	51.6924
0.25	72.9311	27.1130	33.0970	40.3708	49.2055	59.9276
$\to\infty$	96.0000	32.8086	40.8683	50.7593	62.9093	77.8401
		1.1	1.2	1.3	1.4	1.5
Correlation equation for dilatant fluids $f \cdot Re_M = \frac{f \cdot Re}{\beta^*} \times \left\{ \frac{1 + \beta_0 \times (f \cdot Re/f \cdot Re_g)}{1 + \beta_\infty \times (f \cdot Re/f \cdot Re_g)} \right\}$						
1	56.9083	67.8739	80.9241	96.4521	114.926	136.898
0.75	57.9028	69.1804	82.6267	98.6539	117.753	140.504
0.5	62.1922	74.7866	89.8937	108.003	129.697	155.664
0.25	72.9311	88.6790	107.734	130.754	158.514	191.913
$\to\infty$	96.0000	118.736	146.433	180.441	222.178	273.368

$f \times Re_M$ in the high shear rate transition region. For pseudoplastic fluids, the curves have a minimum (maximum for dilatant fluids) located at the midpoint of the $\log(\beta^*)$ range as predicted by Eq. (26). However, the magnitude of $f \times Re_M$ reaches the power law value (i.e., $f \times Re_M = f \times Re_g$) only when the limiting viscosity ratio, β_0/β_∞, is sufficiently large (sufficiently small for dilatant fluids). For practical purposes this occurs when $\beta_0/\beta_\infty \gtrsim 10^4$ for pseudoplastic fluids and when $\beta_0/\beta_\infty \lesssim 10^{-4}$ for dilatant fluids. Thus, the Points "d" and "e" in Fig. 3 are the power law values for $n = 0.5$ and $n = 1.5$, respectively. This implies that care must be exercised when using the power law or modified power law solutions even in those instances where the shear rates throughout the entire flow field are in Region III. If the fluid is weakly pseudoplastic or weakly dilatant, then $f \times Re_M$ may never reach the power law value. For example, consider the flow of a pseudoplastic fluid in a square duct ($\alpha^* = 1.000$, Fig. 3) with a limiting viscosity ratio of $\beta_0/\beta_\infty = 10$ and a flow index of $n = 0.5$. The minimum value of $f \times Re_M$ that this system can achieve is $f \times Re_M = 35.877$ which occurs when the system is operating at $\log(\beta^*) = 0.5$ (Point "f" in Fig. 3). However, the power law value, $f \times Re_g$, for a pseudoplastic fluid with a flow index of $n = 0.5$ is $f \times Re_g = 23.330$ (Point "d" in Fig. 3). Thus, at best the power law solution underestimates the pressure drop by 35% if the system were operating at $\log(\beta^*) = 0.5$, and underestimates it by larger margins if the system is operating elsewhere.

Correlation equations were fit to the computed data, and these are given in Table 1 along with the required constants. The equations are valid within the full range of β^* (i.e., for $\beta^* = 1$ to β_0/β_∞), endpoints excluded. At the endpoints, the expressions correctly approach the Newtonian values as may be verified by performing the limits as β^* approaches both β_0/β_∞ and 1. Values from the correlation equations were compared against the computed numerical data for all duct aspect ratios. Curves from the equations run outside of the computed data with a maximum error of 3.2% for n ranging from 0.70 to 1.30, and 8.0% for $0.50 \leqslant n < 0.70$ and $1.30 < n \leqslant 1.50$. They therefore appear to be satisfactory for most predictive work when $1.30 \leqslant n \leqslant 1.50$ and $0.25 \leqslant \alpha^* \leqslant 1.00$. However, they are not recommended in the case of parallel-plate ducts (i.e., $\alpha^* \to \infty$) since there they produce significantly larger errors, nearly 20% in worst case.

4. Conclusions

The current study details the evaluation of $f \times Re_M$ for flows of hydrodynamically fully developed pseudoplastic and dilatant fluids in rectangular ducts. The analysis utilized constitutive equations that were obtained by combining the MPL models for pseudoplastic and dilatant fluids. This produced equations that extend the valid range of the MPL models to the entire shear rate range while requiring only the limiting Newtonian viscosities, η_0 and η_∞, the power law fluid consistency, K, and the flow index, n, in their definitions. A shear rate parameter was then defined whose value determines the flow regime where the duct is operating. Solving for $f \times Re_M$ throughout the valid range of this shear rate parameter yielded solutions for all operational cases, including when the duct is operating in the high or low shear rate Newtonian regions, in the power law region, or in the low or high shear rate transition regions. The solutions in the high shear rate transition region are particularly important since there appear to be no data published in this regime. Also, as demonstrated above, $f \times Re_M$

values for pseudoplastic fluids reach the power law value only if the ratio of the limiting Newtonian viscosities is sufficiently large (sufficiently small for dilatant fluids). Thus, for weakly non-Newtonian fluids, utilizing the power law values may produce significant errors even in those cases where the shear rates in the flow field are in Region III. Finally, correlation equations are provided that cast the $f \times Re_{\mathrm{M}}$ results in convenient form for numerical work.

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